Infinite Groups and Decision Problems
Background in Algebraic Topology

We follow the books Hatcher (and Lee) as in the course summary.
X is a non empty topological space which is always assumed to be path connected and **locally path connected**:
$\forall x \in X$ and $\forall$ open $U \subseteq X$ with $x \in U$, we have an open path connected set P with $x \in P \subseteq U$.

Let $f, g : X \to Y$ be continuous maps (and let $A \subseteq X$).
A **homotopy** between f and g (relative to A, written rel A) is a continuous map $H : X \times [0,1] \to Y$ with $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$ (with $H(a,t) = f(a)$ $\forall a \in A$ and $\forall t \in [0,1]$, so f and g must agree on A). This is an equivalence relation, written $f \simeq g$ (or $f \simeq g$ rel A).

A **deformation retraction** of X onto $A \subseteq X$ is a homotopy rel A from Id_X to $r : X \mapsto X$ with $r(X) \subseteq A$ and $r|_A = Id_A$.

For $x_0 \in X$ the **fundamental group** $\pi_1(X, x_0)$ is the group of homotopy classes of closed paths γ with start and end x_0 (in other words $\gamma : [0,1] \to X$ is continuous with $\gamma(0) = \gamma(1) = x_0$). Changing the basepoint x_0 "doesn't matter" as we obtain an isomorphic group, written $\pi_1(X)$.

Any continuous map $f : X \to Y$ induces a homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$. If X is homeomorphic to Y (which is also written $X \cong Y$) then $\pi_1(X) \cong \pi_1(Y)$.

X is **homotopy equivalent** to Y (written $X \simeq Y$) if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq \mathrm{Id}_X$ and $fg \simeq \mathrm{Id}_Y$. If so then $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

X is **contractible** if $X \simeq \{x\}$, which implies that X is **simply connected** (meaning that $\pi_1(X)$ is the trivial group).

Seifert - Van Kampen Theorem (Hatcher Theorem 1.20 in the case where the intersections are simply connected): If $X = \bigcup_{\alpha \in A} U_\alpha$ where all of the U_α are path connected and open in X, with the basepoint $x_0 \in \bigcap_{\alpha \in A} U_\alpha$ and each pairwise and triple intersection $U_{\alpha_1} \cap U_{\alpha_2}$, $U_{\alpha_1} \cap U_{\alpha_2} \cap U_{\alpha_3}$ is simply connected then

$$\pi_1(X, x_0) \cong *_{\alpha \in A} \pi_1(U_\alpha).$$

If $A \subseteq X$ and $\iota : A \to X$ is the inclusion map then sadly $\iota_* : \pi_1(A) \to \pi_1(X)$ might not be injective or surjective (see picture). But in a deformation retraction $X \simeq A$ (by taking $f = r$ and g as inclusion of $A = Y$ in X) so X and A have isomorphic fundamental groups.

A continuous surjective map $p : \tilde{X} \to X$ is a **covering map**, with $\tilde{X}$ (which is also taken to be path connected and locally path connected) a **covering space** for X, if $\forall x \in X$ there exists an open neighbourhood V of x with $p^{-1}(V)$ a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically by p onto V. As X is connected, the cardinality of $p^{-1}(x)$ is constant: the **degree** or **number of sheets**.

Given a path $\gamma : [0, 1] \to X$ and a point $\tilde{x}$ above $\gamma(0)$ (this means that $p(\tilde{x}) = \gamma(0)$), there exists a unique path $\tilde{\gamma} : [0, 1] \to \tilde{X}$ lifting γ (namely with $p\tilde{\gamma} = \gamma$). Moreover $p_* : \pi_1(\tilde{X}) \to \pi_1(X)$ is injective.

Now assume X is **locally contractible**.
Classification of Coverings Theorem: For each $H \leq \pi_1(X)$ there exists a cover $p : \tilde{X} \to X$ with $p_*(\pi_1(\tilde{X})) = H$ (this cover is "unique"). The degree is the index of H in $\pi_1(X)$.