Infinite Groups and Decision Problems: Example Sheet 1

Introduction

(**1**) (i) Show that if a group does not have a proper non trivial subgroup then it is (trivial or) isomorphic to C_p for p prime.
(ii) Does every infinite group have infinitely many subgroups? What if we ask the same question but only consider the subgroups up to isomorphism? (What are the easiest subgroups to find in a group?)

(**2**) (This provides another example of a finitely generated group with a subgroup that cannot be finitely generated.)
Consider the subgroup G of $\Sigma(\mathbb{Z})$ generated by $f(z) = z + 1$ and $g = (0\ 1)$, (so g swaps 0 and 1 but fixes everything else).
Show that this contains $S = \Sigma^0(\mathbb{Z})$, which we are defining here to be the subset of permutations of $\mathbb{Z}$ fixing all but finitely many elements.
Show that S is indeed a subgroup of G but cannot be finitely generated.

(**3**) Show (perhaps using the isomorphism theorems) that any subgroup H of $G = \mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^m$ for some $m \leq n$.

Free groups and free products

(**4**) (i) Show that if a group G has a surjective homomorphism θ to a free group F_n of rank n then G can be expressed as a semidirect product $G = K \rtimes F_n$.
(ii) Deduce that the property of **not** containing a non-abelian free group is preserved under subgroups and quotients.

(**5**) (To be used later in the course.)
Suppose we have subgroups $A \leq G$ and $B \leq H$ of the groups G, H, so that we can also regard A, B as subgroups of $G * H$.
Let $S = \langle A, B \rangle \leq G * H$. We will show that S is naturally isomorphic to $A * B$:
(i) Extend the natural inclusions $\iota_A : A \to G * H$, $\iota_B \to G * H$ to a map $\theta : A * B \to G * H$. What is the image of θ?

(ii) Given a non identity element $x \in A * B$ written as

$$x = (a_1)b_1a_2b_2 \dots a_n(b_n)$$

in reduced form, what is $\theta(x)$? Why do we conclude that θ is injective?

(**6**) Given the free product $G * H$ of two groups G and H, show that $G * H$ is finitely generated if and only if both G and H are finitely generated.
Now let $p_H : G * H \to H$ be the unique homomorphism extending the identity from H to H and the trivial homomorphism from G to H.
What is $K = \ker(p_H)$? Is it G?
Show that $G * H$ is also a semidirect product $K \rtimes H$.

(**7**) Use the Klein combination theorem to show that the subgroup of $PSL(2, \mathbb{Z})$ generated by $f(z) = -1/z$ and $g(z) = -1/(z+1)$ is the free product $C_2 * C_3$.

Nielsen-Schreier theorem and index formula

(**8**) (i) For $w(a, b, c) \in F_3$ let $\theta(w)$ be the exponent sum of a in w, that is the number of appearances of a in w minus the number of appearances of a^{-1}.
Show that $\theta : F_3 \to \mathbb{Z}$ is a surjective homomorphism. Let $\theta_k : F_3 \to C_k$ be the exponent sum of a modulo k. What is the rank r of $\mathrm{Ker}(\theta_k)$? Give a set of r generators for $Ker(\theta_k)$.

(ii) What are the index 2 subgroups of F_2?

Presentations of groups

(**9**) (i) (Easy)
Show that the group defined by the presentation

$$\langle a, b | aba^{-1} = b^2, bab^{-1} = a^2 \rangle$$

is trivial.

(ii) (Not quite so easy?)
Do the same for

$$\langle a, b | ab^2a^{-1} = b^3, ba^2b^{-1} = a^3 \rangle.$$