Infinite Groups and Decision Problems:
Example Sheet 2

Presentations of Groups (continued)

(**1**) Given a finite simple (abstract) graph Γ with vertex set $V \neq \emptyset$ and edges E, the *right angled Artin group* (or RAAG) $R(\Gamma)$ is the group defined by the following finite presentation: take one generator for each vertex v of V. Then for each edge $e \in E$ we add the relator $v_i v_j v_i^{-1} v_j^{-1}$, where the endpoints of e are v_i and v_j.
(i) Show that the resulting group $R(\Gamma)$ is always infinite.
(ii) What is the group $R(\Gamma)$ if Γ is the complete graph on k vertices?
(iii) Suppose Γ has connected components $\Gamma_1, \ldots, \Gamma_c$ then how is $R(\Gamma)$ related to $R(\Gamma_1), \ldots, R(\Gamma_c)$?
(iv) Suppose that Γ is *not* a complete graph. Show that the non abelian free group F_2 embeds in $R(\Gamma)$. (Take a quotient by "ignoring" most of the vertices.)

HNN extensions and amalgamated free products

(**2**) (Torsion theorem for HNN extensions)
Suppose that $G*_\phi$ is an HNN extension with base group G and that the element $\gamma \in G *_\phi \setminus G$ has finite order. Write γ in reduced form $g_0 t^{\epsilon_1} \ldots t^{\epsilon_n} g_n$ for $n \geq 1$.
(i) When can we conclude that for any $k > 0$, γ^k is also in reduced form when written out directly?
(ii) If (i) fails then show that there is a element conjugate to γ in $G*_\phi$ having a reduced form of length less than n. Continue this process to conclude that γ must be conjugate in $G*_\phi$ to an element of G. What if G is torsion free?

(**3**) Suppose on forming the amalgamated free product $G *_\phi H$ of groups G, H with isomorphic subgroups $A \leq G$ and $B \leq H$, we actually have B equal to H. What are the A-reduced sequences? What is $G *_\phi H$? What should we take as the definition of a *non trivial* amalgamated free product?

(Lack of) finite index subgroups

(**4**) If G and H are both groups with no proper finite index subgroups then show the same is true of the free product $G * H$. (Look for finite quotients.)

Computability theory

(**5**) *For this question, and all that follow, you may appeal to Church's thesis.*
(i) Show that there are uncountably many sets of integers which are not r.e.
(ii) Using the existence of the halting set $\mathbb{K}$, give an explicit example of a set of integers which is not r.e., and whose complement is not r.e. either.
(iii) Show that every finite set of integers is recursive.

(**6**) Let $\{T_i\}_{i\in I}$ be a collection of Turing machines (on $\{0,1\}$), where we are taking the (fixed) numbering of Turing machines as given in the lectures. Let $X_i := \Omega(T_i)$ be the r.e. set of integers defined by T_i, for each $i \in I$. Show the following:
(i) If I is finite then $\bigcap_{i\in I} X_i$ is r.e.
(ii) If I is r.e. then $\bigcup_{i\in I} X_i$ is r.e.
(iii) (hard) There is an r.e. index set I with X_i *recursive* for all $i \in I$, but for which $\bigcap_{i\in I} X_i$ is not r.e.

The word problem

(**7**) For each of the following, describe a suitable algorithm in your solution:
(i) Show that every finite group has SWP.
(ii) Show that every finitely generated abelian group has SWP.
(iii) Show that every finitely generated free group has SWP.
(iv) Let G, H be recursively presented groups. Show that G, H *both* have SWP $\Leftrightarrow G \times H$ has SWP $\Leftrightarrow G * H$ has SWP.

(**8**) By considering normal forms and Britton's lemma, show the following:
(i) Let G, H be recursively presented groups, and $A \leqslant G$, $B \leqslant H$ be isomorphic finitely generated subgroups. Let $\phi : A \to B$ be an isomorphism. Show that if both G, H have SWP *and* both A, B have solvable MP in G, H respectively, then the amalgamated product $G *_\phi H$ has SWP.
(ii) Let G be a recursively presented group, $A, B \leqslant G$ isomorphic finitely generated subgroups, and $\phi : A \to B$ an isomorphism. Show that if G has SWP *and* both A, B have solvable MP in G, then the HNN extension $G*_\phi$ has SWP.
(iii) Let G be a recursively presented group , and $A \leqslant G$ a finitely generated subgroup. Show that if A has unsolvable MP in G, then we can construct an HNN extension of G with IWP.

Modular machines

(**9**) Let T be a Turing machine in quadruple form with alphabet S and states Q. Give an upper bound, in terms of $|S|$ and $|Q|$, for the number of quadruples of the associated modular machine $\mathcal{M}$, as well as for its modulus m.