Infinite Groups and Decision Problems: Example Sheet 3

Undecidable problems

(**1**) Show that the **conjugacy problem**, of recognising if two words u, v define conjugate elements in the group, is algorithmically unsolvable in some finitely presented group.

(**2**) Show that the **torsion problem**, of recognising if a word w defines an element of finite order in the group, is algorithmically unsolvable in some finitely presented group.

(**3**) Let S be a finitely presented simple group. Show that S has solvable word problem.
Hint: You will need two algorithms: one for when S is the trivial group, and one for when S is non trivial.

Embedding theorems

(**4**) Consider the construction for the Higman embedding theorem, as given in the lectures.
(i) Explain why this construction is **uniformly algorithmic**; that is, there is an algorithm that, on input of any recursive presentation C of a group, outputs a finite presentation P of a group into which $\overline{C}$ embeds.
(ii) Furthermore, explain why this construction gives us an *explicit* embedding (telling us where to send each generator) from the recursive presentation C into the group given by the finite presentation P.

(**5**) Let $o(g)$ denote the order of a group element g, and $\mathrm{Tor}(G)$ denote the set of torsion elements of a group G; $\mathrm{Tor}(G) := \{g \in G \mid o(g) < \infty\}$. Let $\mathrm{TorOrd}(G)$ denote the set of *orders* of non trivial torsion elements of G; $\mathrm{TorOrd}(G) := \{n \in \mathbb{N} \mid 1 < n = o(g) \text{ for some } g \in \mathrm{Tor}(G)\}$.
(i) Show the torsion theorem for amalgamated products: if $G *_\varphi H$ is an amalgamated product, and $\gamma \in G *_\varphi H$, then $\gamma \in \mathrm{Tor}(G *_\varphi H)$ iff γ is conjugate to a torsion element of either G or H. *Hint: Use normal forms and induct.*
(ii) Show that when we embed a countable group G into a 2-generator group G_C as per the Adian-Rabin construction and HNN embedding theorem, we have that $\mathrm{TorOrd}(G) = \mathrm{TorOrd}(G_C)$.
(iii) By considering arbitrary sets of prime numbers S, and the corresponding free product of cyclic groups $*_{p \in S} C_p$ for such sets, show that there are uncountably many 2-generator groups.

(**6**) Show that there exists a finitely presented group into which every *recursively presented* group embeds.

The isomorphism problem

(**7**) Show that there is an algorithm which, on input of a finite presentation P, halts iff $\overline{P} \cong \{e\}$. That is, the set of finite presentations of the trivial group is r.e.

(**8**) Show that there is an algorithm which, on input of two finite presentation P, Q, halts iff $\overline{P} \cong \overline{Q}$. That is, the set of pairs of finite presentations of isomorphic groups is r.e.

(**9**) Show that there is an algorithm which, on input of any two finite presentations P, Q of *free* groups, will always halt and determine whether $\overline{P} \cong \overline{Q}$ or not. That is, the isomorphism problem for finite presentations of free groups is algorithmically solvable.

The Adian-Rabin theorem

(**10**) For the finite presentation $P = \langle x_1, x_2, x_3 \mid r_1, r_2, r_3, r_4 \rangle$ and word w in P, write out the explicit finite presentation $P(w)$ in terms of the x_i's, r_j's, and w (and whatever other additional generators and relators you end up using).

(**11**) Show that each of the following is a Markov property, by giving explicit examples of G_+ and G_- in each case: being trivial, being finite, being free, being abelian, being cyclic, being torsion free, being torsion, having solvable word problem, being simple (make use of question 3 in this final case).

(**12**) Give an algorithm which, on input of a finite presentation P, decides if $\overline{P}$ is **perfect** (that is, if $\overline{P}$ is equal to its derived subgroup; $\overline{P} = [\overline{P}, \overline{P}]$). (*Hint: abelianise*). This is one of the few properties we *can* algorithmically recognise.

(**13**) Show that a finitely presented group G is a universal finitely presented group iff G possesses no Markov property.