PART II AUTOMATA AND FORMAL LANGUAGES
MICHAELMAS 2015-16
EXAMPLE SHEET 2

$*$ denotes a harder problem.

(1) Give decompositions, with proofs, of the integers $\mathbb{N} = A \sqcup B$ into disjoint infinite sets A, B where:

   (a) Both A, B are recursive.

   (b) Both A, B are r.e.

   (c) One of A, B is r.e., the other is not.

   (d) Neither of A, B are r.e.

(2) Give examples, with proofs, of infinite collections of recursive sets whose union:

   (a) Is recursive.

   (b) Is r.e. but not recursive.

   (c) Is not r.e., and its complement is not r.e. either.

(3) Let A be a recursive set, and define the set

$$B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in \mathbb{K}\}$$

Is B recursive? If not, which of B and $\mathbb{N} \setminus B$ are r.e., if any? Provide proofs with your answers.

(4) Construct DFA's, either via transition diagrams or transition tables, which accept precisely the following languages:

   (a) $\{w \in \{0,1\}^* \mid |w| = 2\}$.

   (b) $\{w \in \{0,1\}^* \mid |w| > 2\}$.

   (c) $\{w \in \{0,1\}^* \mid w$ is an alternating sequence of 1's and 0's $\}$.

   (d) $\{w \in \{0,1\}^* \mid w$ is a multiple of 3 when interpreted in binary $\}$.

   (e) $\{w \in \{a, \ldots, z\}^* \mid w = dpmms\}$.

   (f) $\{w \in \{a, \ldots, z\}^* \mid w$ contains $dpmms$ as a substring $\}$.

(5) Construct NFA's, either via transition diagrams or transition tables, which accept precisely the following languages:

(a) $\{w \in \{a, \ldots, z\}^* \mid w$ contains *dpmms* as a substring $\}$.

(b) $\{w \in \{a, \ldots, z\}^* \mid w$ contains *dpmms* and/or *damtp* as a substring $\}$.

(c) $\{w \in \{a, \ldots, z\}^* \mid w \in \{what, where, when\}\}$.

(6) Use the subset construction to convert each of the NFA's you constructed in question 5 to DFA's.

(7) Let L be a regular language over Σ. Prove that the complement of L, $\Sigma^* \setminus L$, is also a regular language over Σ.

(8) Let Σ be a finite alphabet.

(a) By placing an ordering on Σ, give a computable ordering of Σ^*.

(b) Fix some computable ordering of Σ^*, and write w_n for the n^{th} word in this ordering. Show that the function $g : \mathbb{N} \to \Sigma^*$ given by

$$g(n) := w_n$$

is a recursive bijection. That is, given n we can compute the word w_n, and given $v \in \Sigma^*$ we can compute n for which $v = w_n$ as words.

(c) Let L be a regular language over Σ. For any fixed computable ordering of Σ^*, show that

$$\{n \in \mathbb{N} \mid w_n \in L\}$$

is a recursive set.

(9) Prove that the set $\{n \in \mathbb{N} \mid |W_n| > 5\}$ is r.e.

(10*) Prove that the set $\{n \in \mathbb{N} \mid |W_n| > 5\}$ from question 9 is not recursive, without appealing to Rice's theorem.

(11*) Give an example of an infinite collection of recursive sets $\{W_n\}_{n \in I}$, whose index set I is r.e., for which

$$\bigcap_{n \in I} W_n$$

is not r.e.