PART II AUTOMATA AND FORMAL LANGUAGES
MICHAELMAS 2016-17
EXAMPLE SHEET 2

You may appeal to Church's thesis at any time, provided you clearly say so.
$*$ denotes a harder problem.

(1) Give decompositions, with proofs, of the integers $\mathbb{N} = A \sqcup B$ into disjoint infinite sets A, B where:

(a) Both A, B are r.e.

(b) One of A, B is r.e., the other is not.

(c) Neither of A, B are r.e.

(2) Give examples, with proofs, of infinite collections of recursive sets whose union:

(a) Is recursive.

(b) Is r.e. but not recursive.

(c) Is not r.e., and its complement is not r.e. either.

(3) Let A be a recursive set, and define the set

$$B = \{2n \mid n \in A\} \cup \{2n + 1 \mid n \in \mathbb{K}\}$$

Is B recursive? If not, which of B and $\mathbb{N} \setminus B$ are r.e., if any? Prove your answers.

(4) Construct DFA's, via transition diagrams, which accept the following languages:

(a) $\{w \in \{0, 1\}^* \mid |w| > 2\}$.

(b) $\{w \in \{0, 1\}^* \mid w$ is an alternating sequence of 1's and 0's $\}$.

(c) $\{w \in \{0, 1\}^* \mid w$ is a multiple of 3 when interpreted in binary $\}$.

(d) $\{w \in \{a, \ldots, z\}^* \mid w$ contains *ababa* as a substring $\}$.

(5) Construct NFA's, via transition diagrams, which accept the following languages:

(a) $\{w \in \{a, \ldots, z\}^* \mid w$ contains *ababa* as a substring $\}$.

(b) $\{w \in \{a, \ldots, z\}^* \mid w$ contains *dpmms* and/or *damtp* as a substring $\}$.

(c) $\{w \in \{a, \ldots, z\}^* \mid w \in \{what, where, when\}\}$.

Use the subset construction to convert (5a) to a DFA.

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(6) Construct an ϵ-NFA which accepts the *union* of the three languages from question (5), and has only *one* accept state.

(7) Let L be a regular language over Σ. Prove that the complement of L, $\Sigma^* \setminus L$, is also a regular language over Σ.

(8) Let Σ be a finite alphabet, and L be a regular language over Σ.

(a) Fix an ordering of Σ, and use this to describe a well-ordering $\{w_1, w_2, \ldots\}$ of Σ^* by extending the idea of the shortlex ordering of $\mathbb{N}^m$.

(b) Using this ordering of Σ^*, show that $\{n \in \mathbb{N} \mid w_n \in L\}$ is a recursive set.

(9) Prove that the set $\{n \in \mathbb{N} \mid |W_n| > 5\}$ is r.e., but not recursive.

(10) Show that, for each total recursive function $h : \mathbb{N} \to \mathbb{N}$, there is some $n \in \mathbb{N}$ with $f_{n,1} = f_{h(n),1}$ as functions. This is known as the *Recursion Theorem.*
Hint: Use the s-m-n theorem, and a universal partial recursive function.

(11*) Give an example of an infinite collection of recursive sets $\{W_n\}_{n\in I}$, whose index set I is r.e., for which

$$\bigcap_{n\in I} W_n$$

is not r.e.

(12) Consider the set $X = \{n \in \mathbb{N} \mid n \text{ codes a program and } f_{n,1} \text{ is total }\}$.

(a) Show that $\mathbb{K} \leq_m X$, and thus that X is not recursive.

(b) Show that $\mathbb{N} \setminus X$ is not r.e.

(c*) Show that X is not r.e.

(13*) Consider the two sets $A = \{n \in \mathbb{N} \mid n \text{ codes a program and } |W_n| = \infty\}$ and $B = \{n \in \mathbb{N} \mid n \text{ codes a program and } W_n = \mathbb{N}\}$.

(a) With X as in (12), show that $B = X$.

(b) Show that $\mathbb{K} \leq_m A \leq_m B$.

(c) Show that $\mathbb{N} \setminus \mathbb{K} \leq_m A$.

(14) Given an explicit code m for a register machine P_m, consider the set

$$T_m := \{n \in \mathbb{N} \mid n \text{ codes a program and } W_m \subseteq W_n\}$$

Construct two explicit codes m, m' such that T_m is recursive, and $T_{m'}$ is not r.e.
Hint: Use the results of questions (12) *and* (13)*.*

(15*) Let g be a total recursive function on k variables. Show that, with X as in (12), we have $X \leq_m \{n \in \mathbb{N} \mid n \text{ codes a program and } f_{n,k} = g \text{ as functions }\}$. Hence show there is no partial algorithm to verify if answers to question (1) of example sheet 1 are correct, nor a partial algorithm to verify if they are incorrect.