

**PART II AUTOMATA AND FORMAL LANGUAGES**  
**MICHAELMAS 2016-17**  
**EXAMPLE SHEET 3**

\* denotes a harder problem.

- (1) Construct  $\epsilon$ -NFA's and regular expressions for the following regular languages:
- (a) All words  $w \in \{0, 1\}^*$  consisting of either the string 01 repeated some number of times (possibly none), or the string 010 repeated some number of times (possibly none).
  - (b) All words  $w \in \{a, b, c\}^*$  consisting of some number of  $a$ 's (possibly none), followed by some number of  $b$ 's (at least one), followed by some number of  $c$ 's (possibly none).
  - (c) All words  $w \in \{0, 1\}^*$  which contain a 1 somewhere in the last 4 positions. If  $|w| < 4$ , then  $w$  must contain a 1 somewhere.
- (2) Convert each of the following regular expressions to  $\epsilon$ -NFA's:
- (a)  $(0 + 1)(01)$
  - (b)  $(a + bb)^*(ba^* + \epsilon)$
  - (c)  $((aa^*)^*b)^* + c$
- (3) Prove that  $\{w \in \{0, 1\}^* \mid w \text{ contains no more than 5 consecutive 0's}\}$  is regular.
- (4) Let  $R, S, T$  be regular expressions. For each of the following statements, either prove that it is true, or find a specific counterexample.
- (a)  $\mathcal{L}(R(S + T)) = \mathcal{L}(RS) + \mathcal{L}(RT)$
  - (b)  $\mathcal{L}((R^*)^*) = \mathcal{L}(R^*)$
  - (c)  $\mathcal{L}((RS)^*) = \mathcal{L}(R^*S^*)$
  - (d)  $\mathcal{L}((R + S)^*) = \mathcal{L}(R^*) + \mathcal{L}(S^*)$
  - (e)  $\mathcal{L}((R^*S^*)^*) = \mathcal{L}((R + S)^*)$
- (5) Use the pumping lemma to show that none of the following languages are regular:
- (a)  $\{a^n b^n \mid n \geq 0\}$
  - (b)  $\{a^{2n} b^{2n} \mid n \geq 0\}$
  - (c)  $\{ww \mid w \in \{0, 1\}^*\}$

(6) For each of the following languages, determine whether or not they are regular. Justify your answers.

- (a)  $\{a^n b^{2n} \mid n \geq 0\}$
- (b)  $\{a^n b^m \mid n \neq m\}$
- (c)  $\{xcx \mid x \in \{a, b\}^*\}$
- (d)  $\{xcy \mid x, y \in \{a, b\}^*\}$
- (e)  $\{a^n b^m \mid n > m\}$
- (f)  $\{a^n b^m \mid n \geq m \text{ and } m \leq 1000\}$
- (g)  $\{a^n b^m \mid n \geq m \text{ and } m \geq 1000\}$

(7) Prove that no infinite subset of  $\{0^n 1^n \mid n \geq 0\}$  is a regular language.

(8) Find minimal DFA's for each of the following languages. In each case, *prove* that your DFA is minimal.

- (a)  $\{a^n \mid n \geq 0, n \neq 3\}$
- (b)  $\{a^m b^n \mid m \geq 2, n \geq 3\}$
- (c)  $\{a^m b \mid m \geq 0\} \cup \{b^n a \mid n \geq 0\}$

(9) If  $D_1 = (Q, \Sigma, \delta, q_0, F)$  is a minimal DFA, and  $D_2 = (Q, \Sigma, \delta, q_0, Q \setminus F)$  is a DFA for  $\Sigma^* \setminus \mathcal{L}(D_1)$ , then is  $D_2$  necessarily a minimal DFA? Prove your answer.

(10) Let  $L, M$  be languages over  $\Sigma, \Gamma$  respectively. We define the *difference*  $L - M$  to be the words that are in  $L$  but not  $M$ . That is,  $L - M := (L \cup M) \setminus M$ . Show that if  $L, M$  are both regular languages, then  $L - M$  is a regular language over  $\Sigma$ .

(11) Find a regular expression for  $\{w \in \{0, 1\}^* \mid w \text{ is a multiple of 3 when interpreted in binary}\}$ .  
*Hint: Find a suitable DFA, and then convert it to a regular expression.*

(12) Let  $D$  be a DFA with  $N$  states. Prove the following:

- (a) If  $D$  accepts at least one word, then  $D$  accepts a word of length less than  $N$ .
- (b) If  $D$  accepts at least one word of length  $\geq N$ , then  $D$  accepts infinitely many words.

(13) Is the following language regular? Justify your answer.

$$\{1^p \mid p \text{ is a prime}\}$$

(14) Give an algorithm that, on input of a DFA  $D$ , decides if  $\mathcal{L}(D) = \emptyset$  or not.

(15\*) Give an algorithm that, on input of DFA's  $D_1, D_2$ , decides if  $\mathcal{L}(D_1) \subseteq \mathcal{L}(D_2)$  or not.

(16\*) For any  $X \subseteq \{1\}^*$ , show that  $X^*$  is a regular language.